Two and Three parametric regular generalizations of spherically symmetric and axially symmetric metrics

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Abstract. Regular generalizations of spherically and axially symmetric metrics and their properties are considered. Newton gravity law generalizations are reduced for null geodesics.

PACS numbers: 04.20.-q, 04.20.Jb

1. Introduction

According to the Birkgoff theorem [1] *the metric of a vacuum spherically symmetric space time is static, singular in the origin, unique and in spherically symmetric coordinates has the well-known Schwarzschild form* (when $(+,-,-,-)$ signature form is chosen)

$$ds^2 = \left(1 - \frac{a}{r}\right)dt^2 - \frac{dr^2}{1-\frac{a}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1}$$

where $a = 2M$ is the Schwarzschild radius. We define the regularity of a metric at a point as i) an indefinite differentiability of all metric components at this point; ii) the condition $det|g| \neq 0$ holds true at this point relative to any coordinate system associated with our system by a unique and continuous transformation.

Petrov [2] has shown a counterexample of the metric contradicting the Birkgoff theorem

$$ds^2 = \frac{t}{a-t}dt^2 - \frac{a-t}{t}dr^2 - t^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2}$$

where $t < a$. Metric (2) is not static. Here it is necessary to note that (1) and (2) are defined in the same coordinate system. Further, one can find another argument against the uniqueness of (1).The vacuum Einstein equations $R_{ij} = 0$, $i, j = 0, ...3$ are nonlinear second order differential equations. Therefore, according to the general theory

of differential equations[3] one should have at least two constants of integration in the solution, but Schwarzschild metric (1) has only one.

In this Letter we would like to show that it is possible to construct more general metrics in the spherically symmetric and axially symmetric cases. Because of the Letter format we have to restrict the consideration only to the main results without proving the theorems, which we plan to do in the next paper. The question about the applicability of our results in brane models with noncompact extra dimensions and in astrophysics will also be discussed in subsequent publications.

2. Schwarzschild case

Theorem 1. *The spherically symmetric, static and asymptotically free metric, which is a solution of vacuum Einstein equations* $R_{ij} = 0$, $i, j = 0, ...3$ *has the following form:*

$$\begin{aligned} ds^2 \;&=\; \left(1 - \frac{a}{\sqrt[3]{r^3 + b^3}}\right) dt^2 - \frac{r^4 dr^2}{\left(\sqrt[3]{r^3 + b^3}\right)^4 \left(1 - \frac{a}{\sqrt[3]{r^3+b^3}}\right)} \\ &- \left(\sqrt[3]{r^3 + b^3}\right)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (3) \end{aligned}$$

where a *is the Schwarzschild radius and* $b \geq 0$ *is an additional parameter of the metric. This metric is unique to within conformal transformations .*

Conclusion 1. *At* $b = 0$ *we obtain Schwarzschild metric (1).*

Conclusion 2. *At* $b = a$ *we obtain particle- like version (analogously to Ref. [5]) of Schwarzschild metric [4]. It was first obtained by K.Schwarzschild under the condition* $det|g_{ij}| = 1$, *which has violated the general covariance of the theory.*

Thus, metric (3) has the following properties:

(i) when $b = 0$ (the classical Schwarzschild case) it has the curvature singularity at $r = 0$ and the coordinate singularity at $r = a$;

(ii) when $0 < b < a$ it has only the coordinate singularity at $r = \sqrt[3]{a^3 - b^3}$;

(iii) when $b \geq a$ metric (3) is regular everywhere.

The fulfillment of the last contention can be verified by the transformation to the Kruskal-type coordinates

$$v = e^{\frac{R+t}{2a}} \sqrt{\frac{R}{a} - 1}, \qquad u = e^{\frac{R-t}{2a}} \sqrt{\frac{R}{a} - 1},$$

where $R = \sqrt[3]{r^3 + b^3}$. Accordingly, metric (3) takes the form

$$ds^2 = -\frac{4a^3}{R} e^{-\frac{R}{a}} du dv - R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

It is necessary to call the reader's attention to the fact that formally (3) corresponds to all the metric requirements of the differentiable manifold only in the case $b \geq a$. The case $0 \leq b < a$ requires singular coordinate transformations, and that is not completely correct from the formal mathematical point of view while working with smooth manifolds.

Remark. *By chainging radial coordinate r to R metric (3) may be represented in the Schwarzschild form*

$$ds^2 = \Big(1 - \frac{a}{R}\Big)dt^2 - \frac{dR^2}{1 - \frac{a}{R}} - R^2\Big(d\theta^2 + \sin^2\theta d\varphi^2\Big).$$

It seems that parameter b can be cancelled completely, but it would not be correct. In fact parameter b is included into the R domain of definition, namely $R \in [b, \infty)$. For this reason variable R is not equal to the radial coordinate in all the domain of definition. Therefore, any attempt to neglect the parameter b leads to a loss of generality of the results obtained. Moreover, if the Schwarzschild radius $a = 2M$ is defined in such a manner that it corresponds to Newton's law at $r \to \infty$, then, the value of b only changes the geodesics deviations near the origin.

Lemma. *If one has spherically symmetric space time with metric (3), the radial null geodesic equation is*

$$\frac{d^2r}{dt^2} = -\frac{a}{2r^2}\Big(1 - \frac{a}{R}\Big) + \frac{a\ (R-a)^2}{Rr^2}\Big(\frac{1}{2R} + \frac{3}{2}\frac{1}{R-a} - 2\frac{R^2}{r^3}\Big).$$

At $r \to 0$, the asymptotic behavior is:

$$\begin{aligned} \frac{d^2r}{dt^2} &\to -\frac{3}{2}\frac{a^3}{r^4} \qquad \text{when } b = 0, \\ \frac{d^2r}{dt^2} &\to 0 \qquad \text{when } b = a, \\ \frac{d^2r}{dt^2} &\to -2\frac{a\ b\ (b-a)^2}{r^5} \qquad \text{when } b \neq a. \end{aligned} \tag{4}$$

Theorem 1 perhaps improved if one could cancel the condition of staticity. However, the requirement of asymptotical flatness of the metric is essentially necessary due to the Petrov example. Further, the unexpected behaviour of the Newton gravity force near the origin when a particle like version of the metric (with the appropriate choice of the value of parameter b) occurs. The gravity force vanishes at the origin when $b = a$ and tends to infinity as $O(r^{-5})$ at $b > a$. Such unusual behaviour may be responsible for a gravitational collapse without finishing in a black hole configuration [6].

3. Other well-known metrics generalizations

Theorem 2. *The three parametric generalization of Reissner-Nordstrom metric is given*

by

$$ds^2 = \left(1 - \frac{a}{R} + \frac{Q^2}{R^2}\right)dt^2 - \frac{r^4}{R^4}\frac{dr^2}{1 - \frac{a}{R} + \frac{Q^2}{R^2}} - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $R = \sqrt[3]{r^3 + b^3}$ *and* Q *is a U(1) type charge. This metric is regular at the origin when* $b > 0$ *and everywhere when* $b \geq \frac{a+\sqrt{a^2-4Q^2}}{2}$ *and* $a > 2Q$.

Theorem 3. *The three parametric generalization of the Kerr metric is given by*

$$\begin{aligned} ds^2 &= \left(1 - \frac{aR}{R^2 + \beta^2\cos^2\theta}\right)dt^2 - \frac{R^2 + \beta^2\cos^2\theta}{R^2 + \beta^2 - aR}\frac{r^4}{R^4}dr^2 - (R^2 + \beta^2\cos^2\theta)d\theta^2 \\ &- \left(R^2 + \beta^2 + \frac{\beta^2 aR\sin^2\theta}{R^2 + \beta^2\cos^2\theta}\right)\sin^2\theta d\varphi^2 + \frac{a\beta R\sin^2\theta}{R^2 + \beta^2\cos^2\theta}dtd\varphi, \end{aligned}$$

where β *is a specific inertia moment, This metric is regular everywhere when* $b > 0$ *and coordinate singularities are not taken into account.*

4. Conclusions

In this letter the metric that generalizes the well-known Schwarzchild (Reissner-Nordstrom, Kerr) metric is considered . This metric includes (as a partial case) the Schwarzchild metric (when an additional parameter b vanishes), a class of metrics containing only the event horizon without the origin singularity and, finally, a class of singularity free particle-like metrics. The proof of the theorems follows. The applicability of this type of metric rather than the Schwarzchild metric to describe the existing astrophysical data (on accretion, for instance) and to increase the generality of our knowledge of string black holes are under the investigation.

5. Acknowledgments

The author would like to thank S.Alexeyev and all the participants of the Zelmanov Memorial Seminar (http://xray.sai.msu.ru/sciwork/zelmanov) for the useful discussions on the subject of this paper.

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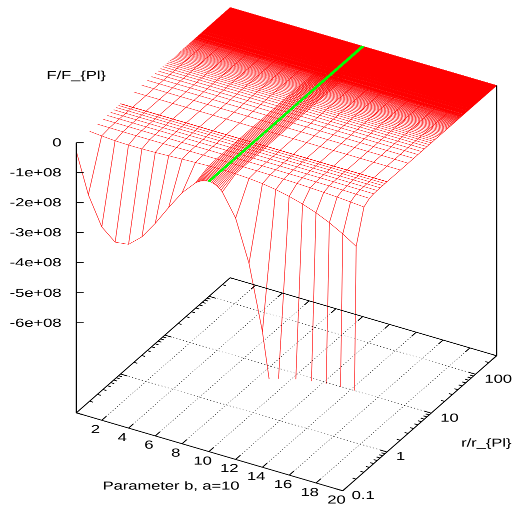

Figure 1. The diagram of gravity force dependency from the radial coordinat r and an additional parameter b for null geodesic in the vicinity of the origin in the case $a = 10$.